Levetidsmodellering: SAINT-modellen

Dansk Demografisk Forening
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Levetidsmodellering: SAINT

Indhold

- Hvad er SAINT?
- Gængse levetidsmodeller
- Dødeligheden i små populationer
- SAINT-modellen
  - trend
  - spread
- Resultater
- Implementering i ATP
ATP’s mortality model

- **SAINT** = Spread Adjusted InterNational Trend
  - describes small population mortality as temporary deviations from underlying trend
  - developed in-house in 2007
- Stochastic mortality model
  - produce a range of possible, future evolutions of mortality intensities, $\mu(t,x)$
  - the mean forecast is used to calculate the tariff and set the reserve for POM
  - calibrated annually
  - no systematic, future increases in reserves due to mortality (if the model is right!)
- Discrete version of SAINT for ATP implemented in P&H
- Continuous version applied to DK developed in academic paper
Mortality models

- Main methodologies
  1. Expert judgment (data free)
  2. Deterministic improvements
  3. Lee-Carter family
  4. Parametric time-series modelling

Deterministic

Stochastic
Mortality modelling

- Lee-Carter (1992)
  - \( \log \mu(t,x) = a(x) + b(x)k(t) + \text{noise} \)
  - assumes age-specific, constant, relative rates of improvement
  - conceptually simple; improvements driven by single index
  - projections overly confident when based only on index variability
  - no structural limitations to the shape of mortality rates; problematic when applied to small population mortality data
  - ”future improvements = historic improvements”;
    the mortality of very old will never improve
  - not very robust; in particular so in small populations
  - various extensions suggested
Mortality modelling

- Parametric time-series modelling
  - assume functional form of (population) mortality, i.e. $\mu(t,x) = F(\theta_t, x)$, e.g. Makeham or logistic period life tables
  - time-series model for (low-dimensional) parameter vector ($\theta_t$)
  - easy to fit and typically provides good description of data
  - provides no insight into what causes the drift in ($\theta_t$)
Forecasting principle

- "In the absence of additional information the best one can do is to extrapolate past trends"
  - sounds sensible, but what does it actually mean?

Model

\[ m(t) = a + b \, t + \epsilon_t \]
\[ m(t)^{1/2} = a + b \, t + \epsilon_t \]
\[ \log m(t) = a + b \, t + \epsilon_t \]
\[ \log m(t) = a + b \, t + c \, t^2 + \epsilon_t \]
Simple projections lack structure and robustness

Danish female mortality

Reasonable short-term projections

Implausible long-term projections lacking (biological) structure
Small population mortality

- Modelling challenge: Produce plausible, long-term forecasts reflecting both the general pattern and the "wildness" seen in data
  - General pattern
    - mortality increases with age
    - age-specific death rates decline over time
    - rates of improvement decrease with age
    - rates of improvement for old age groups increase over time
  - Deviations
    - substantial deviations from the general pattern
    - even periods with increasing mortality for some age groups

- The SAINT model structure

  \[ \text{mortality} = \text{international trend} + \text{spread} \]
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Data and terminology

- Human Mortality Database (www.mortality.org)
- Danish and international female mortality from 1933 to 2005
  - 19 countries in the international dataset: USA, Japan, West Germany, UK, France, Italy, Spain, Australia, Canada, Holland, Portugal, Austria, Belgium, Switzerland, Sweden, Norway, Finland, Iceland & Denmark.
- Death counts and exposures for each year and each age group

\[
D(t,x) = \text{number of deaths} \\
E(t,x) = \text{exposure ("years lived")}
\]

Death rate, \(D(t,x)/E(t,x)\), is an estimate of (the average of) underlying intensity, \(\mu(t,x)\)

Death probability, \(q(t,x) = 1 - e^{-\int_{t}^{t+1} \mu(t,x) dt} \approx \int_{t}^{t+1} \mu(t,x) dt}\)
Danish fluctuations around stable international trend

Danish life expectancy among the highest in the world

Denmark falling behind the international trend

Small improvements at the highest ages

Is this the beginning of a catch up period?
Trend modelling concepts

- **Population dynamics**
  - Ensure consistent intensity surfaces over time and ages by aggregating *individual* intensities to population level
  - Individuals living in the same period of time are influenced by common as well as individual factors
  - Factors have either a cumulative or an instant effect on mortality

- **Frailty (unobservable)**
  - People are genetically different. Only the more robust individuals will attain very high ages
  - Lack of historic improvements among the very old may be due to selection effects. In the future the frailty composition at old ages will change
Homogeneous cohort – no selection

Gompertz-Makeham intensity: $\mu(x) = \alpha e^{\beta x} + \gamma$
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Selection effects within a cohort

Individual: $\mu(x; z) = z\alpha e^{\beta x} + \gamma$

Cohort: $\mu(x) = E(Z \mid x)\alpha e^{\beta x} + \gamma$
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Trend model

- Generalize Gompertz-Makeham intensity to allow for time-dependent cumulative and instant factors

- Underlying individual intensities

\[
\mu(t, x; z) = z \kappa(t) \exp( \int_{t-x}^{t} g(s, s-t+x) \, ds ) + \gamma(t)
\]

\[
\kappa(t) = \exp(\alpha_1 + \alpha_2 (t-t_0)) \quad \text{"treatment" level}
\]

\[
g(t, x) = \beta_1 + \beta_2 (t-t_0) + \beta_3 (x-x_0) \quad \text{"wear-out" rate}
\]

\[
\gamma(t) = \exp(\gamma_1 + \gamma_2 (t-t_0)) \quad \text{"accident" rate}
\]

- mean 1 and variance \( \sigma^2 \) \( \Gamma \)-distributed frailties, \( z \)

- This yields an 8-parameter trend model for population intensity

\[
\mu(t, x) = e^{x-x_0} \kappa(t) \left[ 1 + \sigma^2 \int_{t-x}^{t} e^{x-x} \kappa(u) \, du \right]^{-1} + \gamma(t)
\]

Previous values of \( \kappa \) (and \( g \)) are "remembered" by the population due to selection
Structure of individual intensity

Common and separate components of individual intensities

\[ \mu(t, x_2; z_2) = z_2 \kappa(t) \exp(\int_{t_1}^{t_2} g) + \gamma(t) \]

\[ \mu(t, x_1; z_1) = z_1 \kappa(t) \exp(\int_{t_1}^{t} g) + \gamma(t) \]
Rate of improvement

- Individual and population intensity

\[ \mu(t, x; z) = z \kappa(t) e^{i_x} + \gamma(t) \]

\[ \mu(t, x) = E[Z \mid t, x] \kappa(t) e^{i_x} + \gamma(t) \]

- Senescent component of rate of improvement

\[ \rho_s(t, x) = -\frac{\partial \log(\mu(t, x) - \gamma(t))}{\partial t} = -\frac{\partial \log(E[Z \mid t, x])}{\partial t} - \frac{\partial \log(\kappa(t) e^{i_x})}{\partial t} \]

\[ \rightarrow -\alpha - \beta x \quad \text{for } t \rightarrow \infty \]

- Two opposite effects in rate of improvement:
  - more frail people become old (i.e. first term is negative - and vanishing)
  - general mortality improvements (i.e. second term is positive)
Improvement rates – international trend

Female ($\sigma=0.43$, $\beta_2$ lille)

Male ($\sigma=0.26$, $\beta_2$ stor)
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Trend estimated from international data

- Maximum likelihood estimation based on Poisson-model

\[ D_{\text{INT}}(t, x) \sim \text{Pois}(\mu_{\text{INT}}(t, x)E_{\text{INT}}(t, x)) \]

\[ \mu_{\text{INT}}(t, x) = \left( \mu(t, x) + \mu(t, x+1) + \mu(t+1, x) + \mu(t+1, x+1) \right) / 4 \]

- Estimates \((t_0=2000, x_0=60)\)

<table>
<thead>
<tr>
<th></th>
<th>(\sigma)</th>
<th>(\alpha_1)</th>
<th>(\alpha_2)</th>
<th>(\beta_1)</th>
<th>(\beta_2)</th>
<th>(\beta_3)</th>
<th>(\gamma_1)</th>
<th>(\gamma_2)</th>
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<tbody>
<tr>
<td>Female</td>
<td>4.29e-1</td>
<td>-8.78e0</td>
<td>-1.85e-2</td>
<td>9.90e-2</td>
<td>4.79e-6</td>
<td>1.31e-3</td>
<td>-1.18e1</td>
<td>-8.90e-2</td>
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<tr>
<td>Male</td>
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<td>-1.06e1</td>
<td>-1.78e-2</td>
<td>1.06e-1</td>
<td>8.37e-5</td>
<td>5.59e-5</td>
<td>-7.52e0</td>
<td>-2.50e-2</td>
</tr>
</tbody>
</table>
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Trend – fit and forecast

International female mortality

- Early, young age rate of improvement = 9.1%
- General, long-term rate of improvement = 1.8%
- Increasing old age rate of improvement
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Spread model

- Danish mortality (gender-specific)

\[ \mu_{DK}(t, x) = \mu_{INT}(t, x) \exp(a_t + b_t r_1(x) + c_t r_2(x)) \]

\[ r_1(x) = (x - 60) / 40 \]

\[ r_2(x) = (x^2 - 120x + 9160/3) / 1000 \]

\[ (a_t, b_t, c_t) = A(a_{t-1}, b_{t-1}, c_{t-1}) + e_t, \quad e_t \sim N_3(0, \Omega) \]

- The spread is *assumed* to fluctuate around zero
  - that is, no mean term included in the model

- The spread controls the length and magnitude of deviations
  - and provides information about projection uncertainty

Mean zero, orthogonal regressors normalized to (about) 1 at age 20 and 100
Spread parametrization

Regressors

- Level
- Slope ($r_1$)
- Curvature ($r_2$)
First, we estimate spread parameters \((a_t, b_t, c_t)\) for each year by maximum likelihood based on the Poisson-model:

\[
D_{DK}(t,x) \sim \text{Pois}(\mu_{INT}(t,x) \exp(a_t + b_t r_1(x) + c_t r_2(x)) E_{DK}(t,x))
\]

\[
\mu_{INT}(t,x) = \left(\mu(t,x) + \mu(t,x+1) + \mu(t+1,x) + \mu(t+1,x+1)\right)/4
\]

- trend is kept fixed
- estimates of \((a_t, b_t, c_t)\) depend only on data for year \(t\)

Second, the VAR-parameters \((A, \Omega)\) are estimated based on the estimated time-series of spread parameters \((a_t, b_t, c_t)_{t=1933, \ldots, 2005}\):

\[
(a_t, b_t, c_t) = A(a_{t-1}, b_{t-1}, c_{t-1}) + e_t, \quad e_t \sim N_3(0, \Omega)
\]
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Illustration of spread adjustment

Female mortality in 2004

- International trend
- Danish data
- Danish fit

Estimates
- $a_{2004} = 21\%$
- $b_{2004} = 5\%$
- $c_{2004} = -19\%$
Forecasting

- Forecasting in the VAR-model based on conditional distributions

\[ (a_{T+h}, b_{T+h}, c_{T+h})^\prime | (a_T, b_T, c_T)^\prime \sim N(m_h, V_h) \]

- where \( T \) is the last observation year, \( h \) is the forecasting horizon and

\[
m_h = A^h (a_T, b_T, c_T)^\prime, \quad V_h = \sum_{i=0}^{h-1} A^i \Omega (A^i)^\prime
\]

- Mean forecast

\[
\mu_{DK}(T+h, x) = \hat{\mu}_{INT}(T+h, x) \exp(\bar{a}_{T+h} + \tilde{b}_{T+h} r_1(x) + \bar{c}_{T+h} r_2(x))
\]

- where we use the mean forecast \((\bar{a}_{T+h}, \tilde{b}_{T+h}, \bar{c}_{T+h}) = m_h\)

- trend is kept fixed
Long recovery period

Estimated and forecasted spread

- Fitted $a_t$
- Fitted $b_t$
- Fitted $c_t$
- Forecast
Danish female mortality and international trend

Similar development in old age mortality

Denmark falling behind

... and catching up again
Confidence intervals

- 95%-confidence intervals

\[ \text{CI}_{95\%}(a_{T+h}, b_{T+h}, c_{T+h}) = m_h \pm 1.96 \sqrt{\text{diag}(V_h)} \]

- due to stationarity the variance has a finite limit as \( h \) tends to \( \infty \), i.e. the Danish deviation from the international trend is bounded (in probability)

- 95%-confidence intervals for the intensities

\[ \text{CI}_{95\%}(\log \mu_{DK}(T+h, x)) = \log \mu_{INT}(T+h, x) + m'_h r_x \pm 1.96 \sqrt{r'_h V_h r_x} \]

- where \( r_x = (1, r_1(x), r_2(x))^t \)

- The confidence intervals reflect the stochastic nature of the VAR-model itself

- parameter uncertainty is not taken into account
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Pointwise 95% confidence intervals

Estimated and forecasted spread

Fitted $a_t$, Fitted $b_t$, Fitted $c_t$, Forecast

Year

1950 2000 2050 2100
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Forecast uncertainty

- Analytical methods
  - only feasible for very few quantities of interest, e.g. the spread itself
- Monte Carlo
  - simulate N spread series and calculate mortality forecasts for each
  - calculate quantity of interest, e.g. life expectancy, for each forecast
  - compute uncertainty measures, e.g. 95%-confidence intervals

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Females aged 60 in 2005
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Robustness: Initial year of estimation period

SAINT

Lee-Carter

Log 10 intensity

Year

2010 2020 2030 2040 2050

-3.5 -3.0 -2.5 -2.0 -1.5


x=80 x=70 x=60 x=40

Log 10 intensity

Year

2010 2020 2030 2040 2050

-3.5 -3.0 -2.5 -2.0 -1.5


x=80 x=70 x=60 x=40
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Robustness: Final year of estimation period

SAINT

Lee-Carter

Log 10 intensity against Year

x=80

x=70

www.atp.dk
The SAINT model for ATP

- **ATP mortality (gender specific)**
  \[
  \mu_{ATP}(t, x) = \mu_{INT}(t, x) \exp(a_t^{ATP} + b_t^{ATP} r_1(x) + c_t^{ATP} r_2(x))
  \]
  \[
  \left(a_t^{ATP}, b_t^{ATP}, c_t^{ATP}\right) = A\left(a_{t-1}^{ATP}, b_{t-1}^{ATP}, c_{t-1}^{ATP}\right) + e_t, \ e_t \sim N_3(0, \Omega)
  \]

- The spread parameters are estimated from ATP mortality data
- ATP data only dates back to 1998
- ATP time-series of spread parameters too short to estimate VAR-parameters, we therefore use the Danish VAR-parameters (A and \(\Omega\))
Danish and ATP female spread ($a_t$)
Output

- Cellwise constant mortality surface

\[
\begin{array}{c|c|c}
\text{age} & 63 & 62 \\
61 & \mu(2009,61) & \mu(2010,61) \\
62 & \mu(2009,62) & \mu(2010,62) \\
63 & & \\
\end{array}
\]

1-year survival probability

\[q(n,m) = 1 - \exp(-\mu(n,m))\]
Remaining life expectancy

<table>
<thead>
<tr>
<th></th>
<th>Female</th>
<th></th>
<th>Male</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 years</td>
<td>65 years</td>
<td>0 years</td>
<td>65 years</td>
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<tr>
<td>G82</td>
<td>76.5</td>
<td>17.8</td>
<td>72.7</td>
<td>15.1</td>
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<tr>
<td>ATP2000</td>
<td>79.9</td>
<td>18.4</td>
<td>75.2</td>
<td>15.2</td>
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<td>HMD2000</td>
<td>79.1</td>
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<td>74.4</td>
<td>15.2</td>
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<td>ATP2006</td>
<td>81.0</td>
<td>19.1</td>
<td>76.3</td>
<td>16.7</td>
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<td>HMD2006</td>
<td>80.5</td>
<td>19.0</td>
<td>75.9</td>
<td>16.1</td>
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<td>SAINT.DK (ALDER I 2006)</td>
<td>95.0</td>
<td>21.6</td>
<td>85.0</td>
<td>17.7</td>
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<td>SAINT.ATP (ALDER I 2006)</td>
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<td>95.1</td>
<td>22.7</td>
<td>84.9</td>
<td>17.4</td>
</tr>
</tbody>
</table>
G82M overtaken by reality
Cohort lifetimes

Expected lifetimes of Danish females

SAINT projection

No future improvements

Year

Expected lifetime

Age
The ATP mortality experience for a given year is available for analysis in June the following year.

ATP1: Calculate updated period life table (reserving: age - ½ year)

ATP2: Calculate new spread parameters, e.g. calculate \( (a_{2008}^{ATP}, b_{2008}^{ATP}, c_{2008}^{ATP}) \)

- make new forecast of \( (a_{2009}^{ATP}, b_{2009}^{ATP}, c_{2009}^{ATP}) \), \( (a_{2010}^{ATP}, b_{2010}^{ATP}, c_{2010}^{ATP}) \), …
- make new forecast of entire mortality surface
- trend and VAR-parameters are kept fixed
- if the model is right the annual update will not cause systematic changes in the mortality forecast, nor the reserve

Eventually trend and VAR-parameters should be reestimated.
Forecasts of level parameter \( (a_t) \) for different jump-off years
# Forecasts for remaining life expectancy in 2007 (ATP)

<table>
<thead>
<tr>
<th>Jump-off year</th>
<th>Cohort life expectancy</th>
<th></th>
<th>Period life expectancy</th>
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<tr>
<td></td>
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<td>Male</td>
<td>Female</td>
</tr>
<tr>
<td>0 year</td>
<td>65 year</td>
<td>0 year</td>
<td>65 year</td>
</tr>
<tr>
<td>2003</td>
<td>95.20</td>
<td>21.63</td>
<td>85.13</td>
</tr>
<tr>
<td>2005</td>
<td>95.21</td>
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<tr>
<td>2006</td>
<td>95.21</td>
<td>21.59</td>
<td>85.13</td>
</tr>
<tr>
<td>2007</td>
<td>95.20</td>
<td>21.56</td>
<td>85.12</td>
</tr>
</tbody>
</table>
SAINT in summary

- **Model structure**
  \[
  \text{mortality} = \text{international trend} + \text{spread}
  \]

- **Stable underlying trend**
  - parsimonious parametric model estimated from international data
  - frailty component give rise to changing improvement rates

- **Spread describes the deviations from the trend**
  - stationary, i.e. deviations are effectively bounded
  - allows short- to medium-term fluctuations, but long-term behaviour is determined by the trend